Contaminant migration through fractured till into an underlying aquifer

R. KERRY ROWE

Geotechnical Research Centre, The University of Western Ontario, London, Ont., Canada N6A 5B9

AND

JOHN R. BOOKER

Department of Civil Engineering, The University of Sydney, Sydney 2006, New South Wales, Australia

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This paper examines the potential impact on groundwater quality of contaminant migration from a landfill site, through a fractured till, and into an underlying aquifer. The paper describes a simple, semi-analytic technique for modelling contaminant transport through the fractured till, including consideration of diffusion of contaminants from the fractures into the till matrix, sorption, and radioactive decay. The model also considers the finite mass of contaminant and dilution due to the flow of groundwater in the aquifer. The model can be readily implemented on a microcomputer. The model allows examination of variations in fracture spacing, fracture opening size, thickness of the fractured zone, diffusion coefficient, dispersivity, effective porosity of the matrix, radioactive decay, Darcy velocity, thickness of the aquifer, distribution coefficient, and mass of contaminant. The paper describes the results of a limited parametric study that, *inter alia*, examines the effects of uncertainty in fracture spacing, the thickness of the fractured till, and the effective porosity of the till matrix. Some of the practical implications are discussed.

Key words: contaminant migration, fractures, till, groundwater, pollution, landfill, waste disposal.

Cet article examine l'impact potentiel sur la qualité de l'eau souterraine de la migration à travers un till fracturé, vers la nappe aquifère sousjacente, de contaminants provenant d'un site d'enfouissement. Cet article décrit une technique semi-analytique simple pour modéliser le transport de contaminant à travers le till fracturé, tenant compte de leur diffusion à partir des fractures vers la matrice du till, de la sorption, et de la désintégration radioactive. Le modèle tient compte aussi de la masse finie de contaminant et de la dilution due à l'écoulement de l'eau souterraine dans l'aquifère. Le modèle peut facilement être installé sur un micro-ordinateur. Le modèle permet d'examiner les variations dans l'espacement des fractures, dans la dimension de leur ouverture, dans l'épaisseur de zone de fracture, du coefficient de diffusion, de la dispersivité, de la porosité effective de la matrice, de la désintégration radioactive, de la vélocité de Darcy, de l'épaisseur de l'aquifère, du coefficient de distribution, et de la masse de contaminant. Cet article décrit les résultats d'une étude paramétrique limitée qui, entre autres, examine les effets de l'incertitude dans l'espacement des fractures, l'épaisseur du till fracturé, et la porosité effective de la matrice du till. Certaines des implications pratiques sont discutées.

Mots clés : migration de contaminant, fractures, till, eau souterraine, pollution, enfouissement, entreposage de déchets.

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Introduction

The migration of contaminant through intact clayey soils has been extensively examined in recent years (e.g., see Rowe 1988 for a review of recent research), and natural clayey tills are frequently being proposed as barriers for landfills. Although it is well recognized that the weathered crust in these clayey deposits is often fractured, landfill designers have typically assumed that the underlying unweathered clayey till will not be fractured, and that by locating the base of the landfill beneath the weathered zone, one can avoid the problems that arise from fracturing. Unfortunately, recent research (Herzog and Morse 1986; Ruland 1988; D'Astous *et al.* 1989; Herzog *et al.* 1989) has provided evidence to suggest that till beneath the obviously weathered and fractured zone may be fractured to extensive depths.

For example, the Vandalia till, at a site near Wilsonville, IL, is reported to be weathered to a depth of 5–6 m, underlain by what was originally reported (Johnson *et al.* 1983) to be a generally unfractured, unweathered till about 6 m thick. A more detailed subsequent investigation using angled boreholes indicated that this unweathered–unaltered till was fractured (Herzog and Morse 1986; Herzog *et al.* 1989). Laboratory tests on samples of intact, unweathered till indicated hydraulic conductivities of (3–5) $\times$ 10^{-9} cm/s, while field "slug" tests conducted in conventional vertical boreholes gave a hydraulic conductivity of about 1.3 $\times$ 10^{-7} cm/s. However, field tests conducted in boreholes placed at an angle of 45° gave a hydraulic conductivity of about 1.7 $\times$ 10^{-6} cm/s. The difference between the values from the vertical and angled boreholes was attributed to the fact that the angled boreholes intersected more of the predominantly vertical fractures, which were missed by the vertical holes.

The glacial geology of southern Ontario is such that the clayey tills are frequently underlain by granular aquifers. Thus, the possibility that these clayey tills may be fractured raises the question as to what impact this fracturing would have on contaminant transport through the till and into an underlying aquifer.

Recent investigations in Ontario are consistent with the findings of Herzog and Morse. For example, Ruland (1988) and D'Astous *et al.* (1989) provide evidence to suggest that clayey tills in southern Ontario may be fractured to depths in excess of that evident from conventional boreholes, and to depths below where there is no discernible evidence of weathering in conventional borehole samples. There is

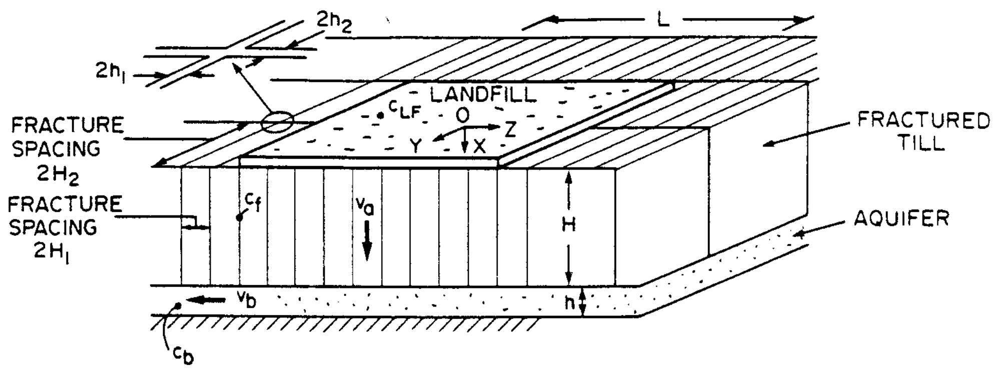


FIG. 1. Problem configuration: fractured till overlying an aquifer.

evidence that these fractures extend to depths in excess of 10 m in some parts of southern Ontario.

Additional evidence of fracturing of "unweathered" tills has been provided by a test pit constructed in Halton till in the town of Milton, Ont. It was reported that the till consisted of a weathered, heavily fractured red–brown clayey silt till to a depth of 3.5 m, underlain by a red–brown clayey silt till with occasional fractures to a depth of 5.1 m, which in turn is underlain by grey clay till with occasional fractures to the bottom of this unit at a depth of 7 m (Eyles 1988).

The recent evidence of fracturing to depths well below the obviously weathered and fractured till has important practical consequences, since approval has been sought, and given, in Ontario for development of a landfill that is separated from an aquifer by about 4 m of unweathered "upper till," even though "there is a possibility that fractures extend through the unweathered upper till..." (Feenstra 1988).

Various investigators have recognized that when considering contaminant transport in fractured porous media, it is essential to consider the coupling of advective–dispersive transport along the fractures with diffusive transport of the contaminant into the porous media adjacent to the fractures (e.g., Foster 1975; Day 1977; Freeze and Cherry 1979; Grisak and Pickens 1980; Grisak *et al.* 1980; Barker and Foster 1981; Tang *et al.* 1981; Sudicky and Frind 1982).

Finite-element techniques provide one means of modelling contaminant migration in fractured systems (e.g., Grisak and Pickens 1980; Huyakorn *et al.* 1983; and others). These approaches potentially make it possible to analyze complicated two- and three-dimensional fracture networks for complex boundary conditions. However, these approaches are most useful when there are detailed data available concerning the distribution and characteristics of the fracture system. Frequently, these data are not available, and it is necessary to assess potential impact based on a knowledge of typical fracture spacings and orientations, together with some knowledge of the hydraulic characteristics of the system (i.e., hydraulic gradient and hydraulic conductivity). Under these circumstances, analytic or semi-analytic solutions for contaminant transport in a fractured medium may be particularly useful for quickly assessing the potential effects of uncertainty regarding key parameters (e.g., fracture spacing, fracture opening size, etc.). These techniques may also be useful for benchmarking more complex numerical procedures (e.g., finite element codes).

Several investigators have developed analytical or semi-analytical solutions for contaminant transport in idealized fracture media. For example, Neretnieks (1980) and Tang *et al.* (1981) developed similar solutions for one-dimensional (1-D) contaminant transport along a single fracture together with 1-D diffusion of contaminant into the matrix of the rock adjacent to the fracture. Sudicky and Frind (1982) and Barker (1982) extended this approach to consider the case of multiple parallel fractures. The aforementioned researchers all considered constant concentration at the inlet of the fractures. Moreno and Rasmuson (1986) developed an analytical equation for a constant-flux boundary condition at the inlet of a single fracture.

Germain and Frind (1989) and Sudicky (1989) have developed a technique for the analysis of contaminant transport through fractured media that involves the use of the finite-element technique to make the space domain discrete; it uses the Laplace transform technique in the time domain. This approach provides more flexibility than semi-analytic methods while sidestepping some of the numerical problems associated with the conventional use of finite-element methods.

Rowe and Booker (1988, 1989*a*, 1990) developed a more general form of the semi-analytic solutions noted above. They produced solutions for 1-D, 2-D, and 3-D fractured systems that allow consideration of the finite mass of contaminant within a landfill as well as consideration of 1-D, 2-D, or 3-D diffusion into the matrix of blocks from the fractures. However, like all the preceding semi-analytic solutions, these solutions also considered the fractured mass to be of infinite extent. This may represent a reasonable approximation for lateral migration from a landfill, but this does not represent a reasonable approximation of the situation where a landfill is separated from an underlying aquifer by a relatively thin (say 2–10 m) thick layer of fractured till.

In principle, the migration of contaminant transport through a fractured barrier and into an underlying aquifer could be modelled using conventional finite-element techniques. However, this approach involves considerable numerical difficulties and is not well suited to the conventional landfill design. Thus, the objective of this paper is to outline a semi-analytic technique that allows a typical designer to obtain an approximate solution to this problem of migration through a thin fractured barrier in a few seconds using a microcomputer. A second objective is to perform a limited parametric study to provide the designer with an indication of the potential effects of fracture flow on contaminant transport and, hence, impact within an underlying aquifer, and to compare results obtained for the same bulk hydraulic conductivity assuming (*a*) no fractures and (*b*) fractures. The application section is focused on fractured till. However, the theory presented in the following section can be used equally well for other fractured porous media (e.g., fractured clay).

It should be emphasized that this paper is concerned with the migration of contaminants typical of that encountered in domestic waste leachate. The migration of concentrated dense, nonaqueous phase liquids is beyond the scope of this paper.

Theory

Consider an extensive landfill that has its base resting on fractured clayey till. Suppose that there are two possible sets of orthogonal planar fractures in the till. Referring to Fig. 1, set 1 is parallel to the 0-y coordinate axis and is spaced a distance $2H_1$ apart with a fracture opening size of $2h_1$. Set 2 is parallel to the 0-z coordinate axis and is at a spacing of $2H_2$ with a fracture opening size of $2h_2$.

It is assumed that contaminant transport from the landfill is by advective–dispersive transport through the fractures in the fractured till. The fractured till, of thickness H, is underlain by an aquifer of thickness h. The downward Darcy velocity (i.e., flow per unit area) through the fractured till is v_a. The horizontal Darcy velocity in the aquifer (at the downgradient edge of the landfill) is v_b.

Based on considerations of conservation of mass, one can show that the partial differential equation governing contaminant transport through a unit volume of the fractured till is given by

$$[1]\quad D_a \frac{\partial^2 c_f}{\partial x^2} - v_a \frac{\partial c_f}{\partial x} = (n_f + \Lambda K_f)\left(\frac{\partial c_f}{\partial t} + \lambda c_f\right) + \dot{q}$$

where c_f is the increase in contaminant concentration in a representative fracture at a depth x and time t;

$$[2a]\quad n_f = \frac{h_1}{H_1} + \frac{h_2}{H_2}$$

is the fracture porosity perpendicular to the flow direction;

$$[2b]\quad D_a = D_{1x}\frac{h_1}{H_1} + D_{2x}\frac{h_2}{H_2}$$

and

D_{1x} and D_{2x} are the coefficients of hydrodynamic dispersion in fracture sets 1 and 2 respectively;

$$[2c]\quad v_a = v_{1x}\frac{h_1}{H_1} + v_{2x}\frac{h_2}{H_2}$$

is the downward Darcy velocity and v_{1x} and v_{2x} are the groundwater (fracture) velocities in fracture sets 1 and 2 respectively;

$$[2d]\quad \Lambda = \frac{1}{H_1} + \frac{1}{H_2}$$

represents the surface area of the fracture per unit volume of till; K_f is the fracture distribution coefficient, defined by Freeze and Cherry (1979) as the mass of solute adsorbed per unit area of surface divided by the concentrations of solute in solution; λ is the decay constant of the solute (i.e., ln 2 divided by the solute's mean half life). The function λc_f may result from radioactive decay, biological degradation, or chemical reactions among different species in solutions; $\dot{q}$ is the rate at which the contaminant is being transported into the matrix (per unit volume) by diffusion from the fractures.

For the special case where both fracture sets are identical, $D_{1x} = D_{2x} = D$ and $v_{1x} = v_{2x} = v$, hence,

$$[3a]\quad D_a = n_f D$$

$$[3b]\quad v_a = n_f v$$

where D is the coefficient of hydrodynamic dispersion in the fractures, and v is the groundwater velocity (fracture velocity).

Equation [1] must be solved subject to appropriate boundary conditions. First, consider the landfill. Suppose that at the time of completion of the landfill there is a finite mass of contaminant m_{TC} in the landfill. This mass can be represented in terms of a "representative height of leachate" H_r.

$$[4a]\quad H_r = \frac{m_{TC}}{A_o c_o}$$

where A_o is the plan area of landfill through which contaminant can escape. (Thus, m_{TC}/A_o is the mass of contaminant per unit area). c_o is the peak concentration of contaminant in the landfill.

Now suppose that the infiltration into the landfill is q_o and that the exfiltration through the base of the landfill is q_a (v_a); then, to sufficient accuracy, the proportion of the mass available for transport into the underlying till is given by the "equivalent height of leachate" H_f (see Rowe 1988 for more detail); viz.,

$$[4b]\quad H_f = \frac{q_a}{q_o} H_r$$

It is noted that the use of the equivalent height of leachate, H_f, avoids the need to directly model any leachate collection system. An alternative approach, which uses the representative height of leachate, H_r, directly and which explicitly models contaminant loss to the leachate collection system, has been described by Rowe and Booker (1989*b*). For the type of case considered here, the end result is, for all practical purposes, identical.

When contaminant transport is primarily through the fractures (rather than the matrix), as assumed here, conservation of mass gives rise to the following simple relationship at the boundary between the landfill and the till:

$$[5]\quad \left(\frac{\partial c_{LF}}{\partial t} + \lambda c_{LF}\right) + \left.\frac{F_x}{H_f}\right|_{x=0} = 0$$

where $c_{LF} = c_o$ at $t = 0$ is the concentration in the landfill for $t \geq 0$.

$$[6]\quad F_x = v_a c_f - D_a \frac{\partial c_f}{\partial x}$$

is the mass flux of contaminant in the x direction.

Next, consider the concentration c_b in the underlying permeable layer of thickness, h (see Fig. 1). It is assumed that the landfill is of width L and that the lateral Darcy velocity at the down-gradient edge of the landfill is v_b. Now suppose that transport within the permeable layer is purely advective; then, a simple equation defining the concentration in the aquifer can be determined (e.g., see Rowe and Booker 1985*a*, 1987) as follows:

$$[7]\quad hL\left(\frac{\partial c_b}{\partial t} + \lambda c_b\right) - LF_x\Big|_{x=H} + h v_b c_b = 0$$

Equations [1] and [5]–[7] govern the migration of contaminant from the landfill, through the fractured till, and into the underlying aquifer. These equations can be readily solved by introducing a Laplace transform; viz.,

$$[8a] \quad \bar{c}_f = \int_0^\infty e^{-st} c_f(t)\, dt$$

$$[8b] \quad \bar{F}_x = \int_0^\infty e^{-st} F_x(t)\, dt$$

Thus, taking the Laplace transform of [1] gives

$$[9] \quad D_a \frac{\partial^2 \bar{c}_f}{\partial x^2} - v_a \frac{\partial \bar{c}_f}{\partial x} = (n_f + \Lambda K_f) + (s + \lambda)\bar{c}_f + \bar{q}$$

where s arises from taking the Laplace transform. The term $\bar{q}$ can be determined for a 1-D, 2-D, or 3-D block of matrix material. As shown by Rowe and Booker (1990), $\bar{q}$ can be written in the form

$$[10] \quad \bar{q} = (s + \lambda)\bar{\eta}\, \bar{c}_f$$

where $\bar{\eta}$ is given in the Appendix for 1-D and 2-D conditions. Substituting [10] into [9] then gives

$$[11] \quad D_a \frac{\partial^2 \bar{c}_f}{\partial x^2} - v_a \frac{\partial \bar{c}_f}{\partial x} = [(n_f + \Lambda K_f) + \bar{\eta}](s + \lambda)\bar{c}_f$$

This has a solution of the form

$$[12] \quad \bar{c}_f = Ae^{-\alpha x} + Be^{-\beta x}$$

$$[13] \quad F_x = -D_a[\beta Ae^{-\alpha x} + \alpha Be^{-\beta x}]$$

where A and B are constants to be determined from the boundary conditions and $k = \alpha$ and β are the solutions of the quadratic equation:

$$[14a] \quad D_a k^2 + v_a k - [(n_f + \Lambda K_f) + \bar{\eta}](s + \lambda) = 0$$

viz

$$[14b] \quad \alpha,\beta = \frac{-v_a}{2D_a} \pm \frac{\sqrt{v_a^2 + 4D_a[(n_f + \Lambda K_f) + \bar{\eta}](s + \lambda)}}{2D_a}$$

The constants A and B can be evaluated from the condition that the concentration in the till must be the same as that of the landfill at their interface ($x = 0$), and the concentration in the till must be identical to that in the underlying permeable layer at their interface ($x = H$).

Taking the Laplace transform of the surface boundary condition (eq. [5]) and noting that $c_{LF} = c_f$ when $x = 0$ gives

$$[15] \quad (s + \lambda)\, \bar{c}_f + \frac{\bar{F}_x}{H_f} = c_o \text{ when } x = 0$$

and thus, using [12] and [13], it is found that

$$[16] \quad A\left[1 - \frac{\beta D_a}{(s + \lambda)H_f}\right] + B\left[1 - \frac{\alpha D_a}{(s + \lambda)H_f}\right] = \frac{c_o}{(s + \lambda)}$$

Similarly, taking the Laplace transform of the boundary condition at the aquifer (eq. [7]) gives

$$[17] \quad (s + \lambda)hL\, \bar{c}_b - L\bar{F}_x\Big|_{x = H} + hv_b\, \bar{c}_b = 0$$

Now recalling that $c_b = c_f$ when $x = H$, it follows that

$$[18] \quad \bar{c}_f\left(1 + \frac{v_b}{(s + \lambda)L}\right) - \frac{\bar{F}_x}{(s + \lambda)h} = 0 \text{ when } x = H$$

and thus, using [12] and [13], it is found that

$$[19] \quad A\left[\left(1 + \frac{v_b}{(s + \lambda)L}\right) + \frac{D_{ax}\beta}{(s + \lambda)h}\right]e^{-\alpha H} + B\left[\left(1 + \frac{v_b}{(s + \lambda)L}\right) + \frac{D_{ax}\alpha}{(s + \lambda)h}\right]e^{-\beta H} = 0$$

The unknown constants A and B can now be evaluated by solving [16] and [19], and it is found that

$$[20] \quad A = \frac{c_o}{(s + \lambda)} \frac{\left(1 + \frac{v_b}{(s + \lambda)L} + \frac{D_{ax}\alpha}{(s + \lambda)h}\right)e^{-\beta H}}{\Delta}$$

$$B = -\frac{c_o}{(s + \lambda)} \frac{\left(1 + \frac{v_b}{(s + \lambda)L} + \frac{D_{ax}\beta}{(s + \lambda)h}\right)e^{-\alpha H}}{\Delta}$$

where

$$\Delta = \left[1 - \frac{\beta D_a}{(s + \lambda)H_f}\right]\left[1 + \frac{v_b}{(s + \lambda)L} + \frac{D_a\alpha}{(s + \lambda)h}\right]e^{-\beta H} - \left[1 - \frac{\alpha D_a}{(s - \lambda)H_f}\right]\left[1 + \frac{v_b}{(s + \lambda)L} + \frac{D_a\beta}{(s + \lambda)h}\right]e^{-\alpha H}$$

Equations [12], [14*b*], and [20] now completely define the solution in Laplace transform space. The solutions in the time domain can now be determined using numerical inversion; for example, using the method proposed by Talbot (1979).

Applicability of approach

The theory presented in the previous section models the aquifer in the same way that it was modelled in a previous formulation developed by the authors (Rowe and Booker 1985*a*, 1987). The primary difference between this and the earlier formulation is that in the present approach, the barrier unit (till) is modelled as a fractured porous media, whereas application of the earlier (Rowe and Booker 1985*a*) formulation involves modelling of the barrier unit (till) as an intact porous medium.

It is assumed here that there is 1-D contaminant transport along the fractured system coupled with 2-D diffusion from the fractures into the matrix. The model calculates the total mass balance within the aquifer directly beneath the landfill. Thus, for any given time, the mass of contaminant in the base aquifer is obtained by calculating the total mass into the aquifer due to 1-D advective–dispersive transport from the entire width of the landfill and subtracting the mass that has been removed from the aquifer beneath the landfill by advective transport down-gradient of the landfill. The base

TABLE 1. Summary of parameters considered

Quantity	
Width of landfill, L (m)	200
Equivalent height of Leachate, H_f (m)	1
Initial concentration, c_o (mg/L)	1500
Downward Darcy velocity, v_a (m/a)	0.05
Thickness of fractured clay, H (m)	4, 10
Porosity of clay matrix, n_m	0.25, 0.4
Thickness of underlying aquifer, h (m)	1
Porosity of aquifer, n_b	0.3
Horizontal Darcy velocity in aquifer, v_b (m/a)	11
Diffusion coefficient in matrix, D_m (m^2/a)	0.02
Retardation coefficient for matrix, R_m	1
Coefficient of hydrodynamic dispersion along fractures, D (m^2/a)	0.06
Fracture spacing, $2H_1 = 2H_2$ (m)	0.5,1,1.5,2,2.5,3,3.5,4,5
Fracture opening size, $2h_1 = 2h_2$ (μm)	4,5.5,6,6.5,7,8,8.3,8.5,9

concentration is then this mass of contaminant divided by the volume of fluid in the aquifer beneath the landfill.

For any given time, a single base concentration, c_b, is determined, and this provides an estimate of the concentration in the aquifer at the down-gradient edge of the landfill. Since the concentration is averaged over the entire thickness of the aquifer, care is required in practical applications to use an aquifer thickness in the analysis that the user considers would reasonably reflect the thickness over which mixing (dilution) of contaminant would occur.

In cases of uncertainty, one can perform a sensitivity study varying the thickness, h (see Fig. 1), to determine the implications of uncertainty regarding the mixing thickness. All other factors being equal, the smaller the mixing thickness h, the higher the calculated base concentration, since there is less dilution of contaminant in the aquifer. This is only likely to be a major consideration for thick aquifers. Given that wells are likely to be screened across the full thickness of thin aquifers, it would be reasonable to use the full thickness in many such situations. Clearly, however, each hydrogeologic situation has its own characteristics, and all parameters for use in any analysis (including the thickness h being discussed here) must be selected by a suitably qualified hydrogeologist or hydrogeotechnical engineer on a site-specific basis.

The model assumes that all contaminant enters the till through the fractures, and the model does not consider diffusion directly from the landfill into the till matrix. As a consequence, the model should not be applicable when the fracture spacing is substantially greater than the thickness of the fractured deposit or when the fractures are so tight that migration is through the matrix rather than the fractures. Thus, in many practical applications, it would be appropriate to model the problem in two distinct analyses, viz., (*a*) considering advective–diffusive transport through the matrix (e.g., as discussed by Rowe and Booker 1985*a*) and (*b*) considering migration along fractures and matrix diffusion (i.e., the theory in this paper). Engineering judgement would then be used in selecting the most relevant results from these two analyses.

Results

To illustrate the application of the theory, consider a hypothetical landfill 200 m wide (i.e., $L = 200$ m) separated from a 1-m-thick sand aquifer by fractured clayey till of thickness H, as shown in Fig. 1. The parameters defining the problem are summarized in Table 1. The parameters selected in this example represent a rather severe situation, where there is a relatively high (for a clayey barrier) downward Darcy velocity and very little dilution in the underlying aquifer. If the downward velocity was smaller (or the flow in the aquifer was greater) than assumed here, then the impacts would be less than that indicated in the following discussion and Figs. 2–7. It should also be emphasized that this example is for illustration of the potential application of the theory. The results presented are for a specific set of parameters and should not be generalized beyond the range of parameters considered. When dealing with fractured flow, there is a complex symbiotic relationship between the mass of contaminant in the landfill, the migration along the fractures, and attenuation into the matrix, which can only be adequately considered on a site-specific basis for ranges of parameters that are reasonable for that particular situation. The interaction between these parameters can be modelled using the theory presented in this paper.

In all of the following discussion, it is assumed that there are two orthogonal fracture sets with the same spacing ($2H_1 = 2H_2$). However, the theory readily permits consideration of other cases, including either a single fracture set or two orthogonal sets with different spacings (i.e., $2H_1 \neq 2H_2$). It should also be noted that the fracture spacing refers to the spacing between fractures, which provides a continuous path from the landfill to the underlying aquifer; dead-end fractures are ignored, since they will not provide a conduit for significant contaminant transport.

For the assumed conditions, involving a 4-m thickness of fractured clay with a matrix porosity of 0.25, the calculated variations in concentration with time in the aquifer are summarized in Fig. 2 for fracture spacings of 1, 2, and 5 m. For the purposes of this comparison, it is assumed that the

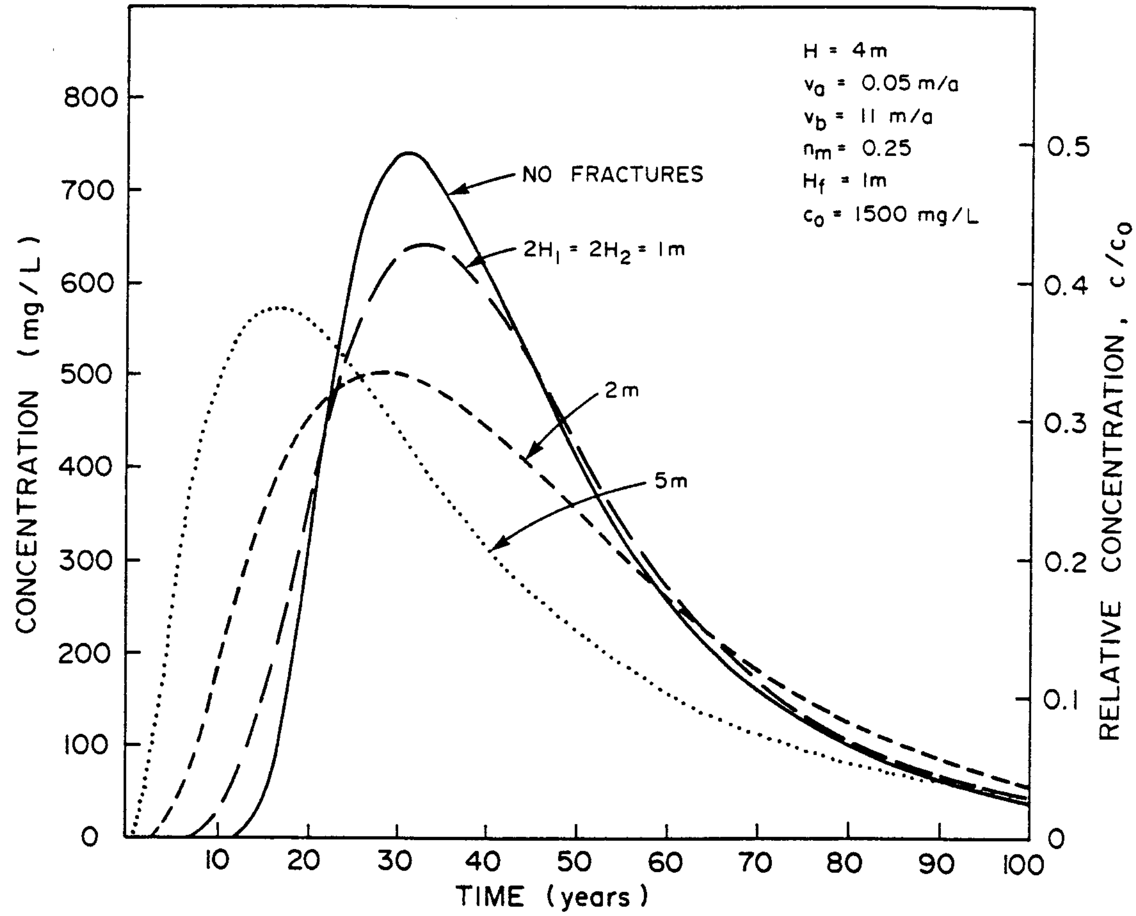


FIG. 2. Variation in concentration in the aquifer with time assuming an unfractured and a fractured till separating a landfill from an underlying aquifer. (See Table 1 for parameters not given on the figure.)

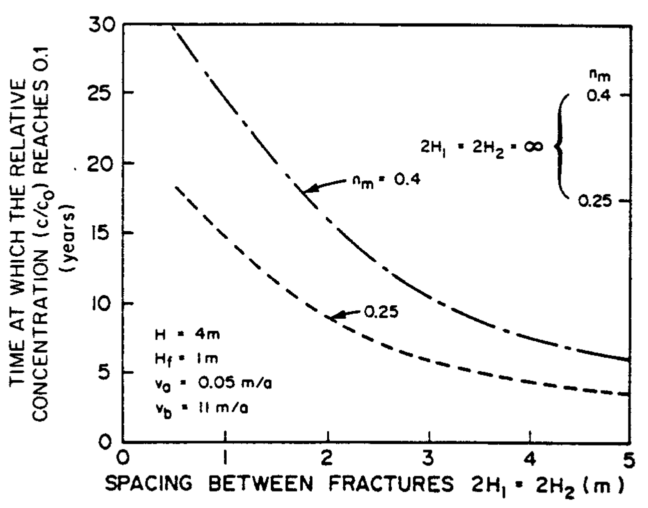


FIG. 3. Time to reach 10% of source concentration in an aquifer as a function of fracture spacing.

bulk hydraulic conductivity is known (e.g., from a field pump test) and that the major uncertainty relates to the fracturing, which gives rise to this bulk hydraulic conductivity. Thus, the variation in fracture spacing considered in Fig. 2 involves a corresponding variation in fracture opening size (using a relationship from Hoek and Bray 1981) such that the hydraulic conductivity remains constant. Figure 2 also shows results for the special case of advective–diffusive transport through the matrix alone (i.e., no fractures).

Examining the results shown in Fig. 2, one can see that even if the bulk hydraulic conductivity is known, variation in the fracture spacing (and opening size), which could give rise to this hydraulic conductivity, has a significant effect on the arrival of contaminant in the aquifer. This is not, in itself, surprising since increasing the fracture spacing reduces the fracture porosity, which in turn, increases the groundwater velocity through the fractures. What is of note is that the time of impact on the aquifer can be quantified (allowing for dilution effects in the aquifer) using this proposed technique and that relatively widely spaced fractures (which would be difficult to detect in a conventional investigation using vertical boreholes) can have a substantial effect on contamination of the aquifer beneath a proposed landfill.

The reasonable-use guideline of the Ministry of the Environment (MOE) in Ontario (MOE 1986) limits the increase in contaminant concentration, due to a landfill, within a groundwater resource. The limit on the increase in concentration for chloride is 125 mg/L, and lesser increases are permitted if there is an existing level of chloride in the aquifer. Thus, if the initial source concentrations of chloride in the landfill were, say, 1250 mg/L, the impact on an underlying aquifer would exceed the MOE limit when the increase in concentration, c, in the aquifer had increased to 10% of the source value, c_o (i.e., $c/c_o = 0.1$). Figure 3 shows the time required to exceed MOE limits assuming zero background chloride concentration and a source concentration of 1250 mg/L for a range of fracture spacings.

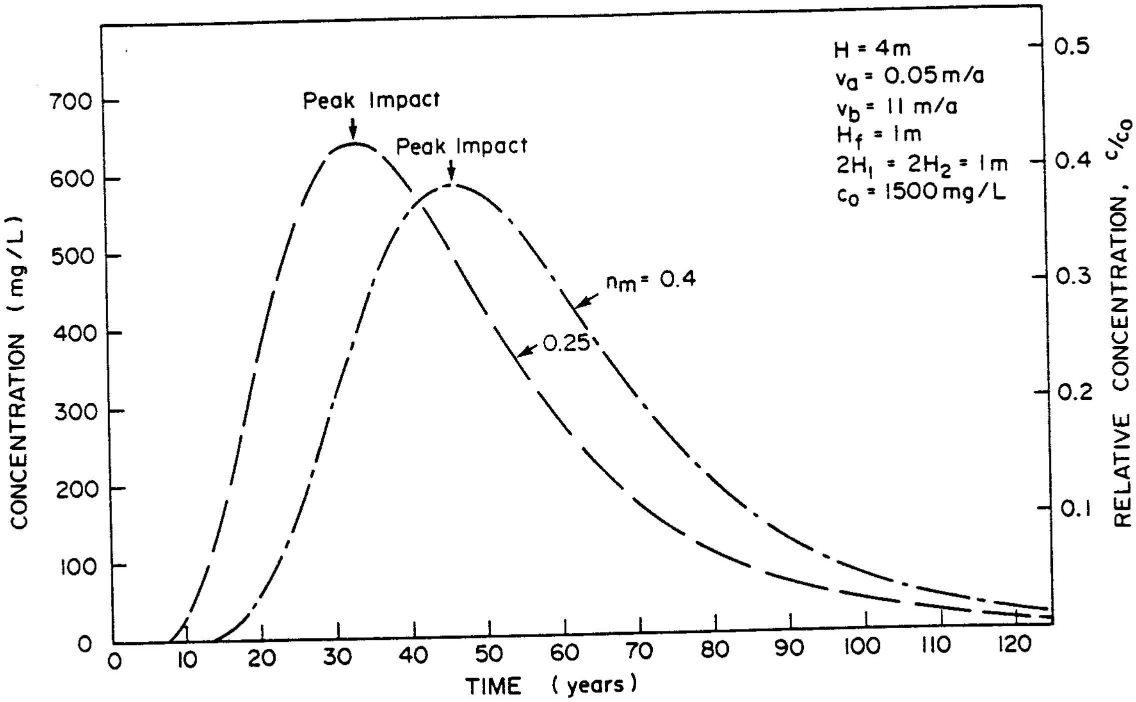


FIG. 4. Variation in concentration in an aquifer with time for two assumed values of the effective porosity of the till matrix.

The results presented in Fig. 3 for a matrix porosity of 0.25 correspond to the same case for which the variation is calculated with time and given in Fig. 2. It is evident from this figure that an impact exceeding allowable limits could occur in a relatively short period of time if the till is fractured. This has important implications for monitoring programs, since it means that there is a much shorter time in which to establish the monitors (i.e., boreholes in the aquifer that monitor groundwater quality) and background data than would be expected if the till were not fractured. In addition, inspection of Fig. 2 shows that once contaminant reaches the aquifer, the increase in concentration with time is very rapid. This means that once the problem is detected, action must be taken quickly to prevent off-site migration; one does not have the luxury of many years between the time of first detection and major contamination of the aquifer for a scenario similar to that being examined here.

The actual time at which excessive impact occurs for a given level of fracturing depends on the diffusion coefficient and sorption in the clay matrix, and on the effective porosity of the clay matrix. Figure 4 shows the effect of matrix porosity on the variation of the concentration in the aquifer with time for a fracture spacing of 1 m. It can be seen that the higher the matrix porosity, the longer it takes for contaminant to reach the aquifer in significant quantities. This situation arises because the higher porosity allows a greater mass of contaminant to migrate from the fractures into the matrix and, hence, by removing contaminant from the fractures, decreases the mass arriving at the aquifer at early times. For the same reason, the peak impact on the aquifer is reduced, and the time at which this peak is reached is increased.

If the concentration in the landfill decreases with time, eventually there will be a point in time where the concentration of the leachate entering the fracture system is less than the concentration of contaminant that is "stored" in the pores of the clay matrix. Once this occurs, contaminant will diffuse out of the clay back into the fractures. The higher the effective matrix porosity (all other things being equal), the greater the mass of contaminant available to diffuse back into the fractures. This effect is evident from Fig. 4, where it can be seen that at longer times, the concentration in the aquifer calculated for a matrix porosity of 0.4 exceeds that calculated for a matrix porosity of 0.25.

The results presented in Fig. 4 are for a specific fracture spacing; however, the general effects of matrix porosity are similar for all fracture spacings. For example, Fig. 3 shows the time at which the concentration in the aquifer reaches 10% of the initial leachate value (i.e., $c/c_o = 0.1$) for a range of fracture spacings and matrix porosities of both 0.25 and 0.4. It is evident that for each fracture spacing, the time at which this impact is reached is greater for a porosity of 0.4 than for a porosity of 0.25. Clearly, a reasonable estimate of matrix porosity is required if reasonable estimates of impact are to be made.

The results presented in Figs. 2–4 were for a 4-m thick fractured layer. Figure 5 shows the variation in concentration with time for an aquifer if it is separated from the landfill by a 10-m-thick fractured layer. Comparison of Figs. 2 and 5 indicates that for a given fracture spacing, increasing the thickness of the deposit increases the time required for a given impact to be reached in the aquifer (all other things being equal) and decreases the maximum impact. Both of these trends can be explained by the fact that for a thicker deposit, there is a greater volume of matrix pores into which contaminant can diffuse as it moves along the fractures.

Examination of Figs. 2 and 5 shows that for each combination of parameters considered, the concentration of contaminant within the aquifer increases to a peak value and then subsequently decreases as contaminant is "washed" out of the system. This general trend is a consequence of there being a finite mass of contaminant within the landfill. The magnitude and time of occurrence of this peak depend among other things, on the mass of contaminant available

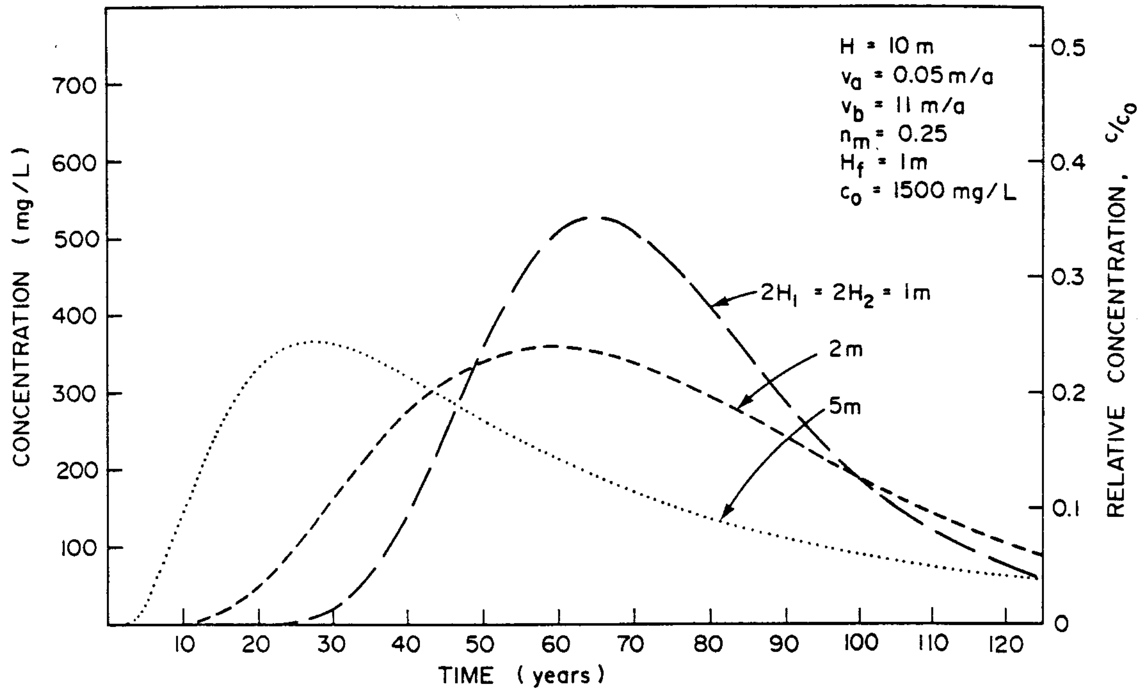


FIG. 5. Variation in concentration in the aquifer with time for a 10-m-thick fractured layer.

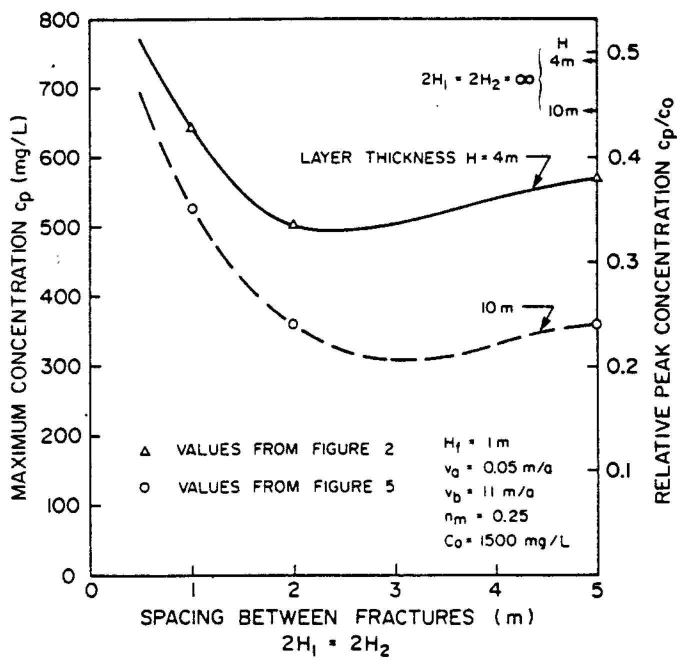


FIG. 6. Variation in peak impact on groundwater quality in an aquifer with fracture spacing.

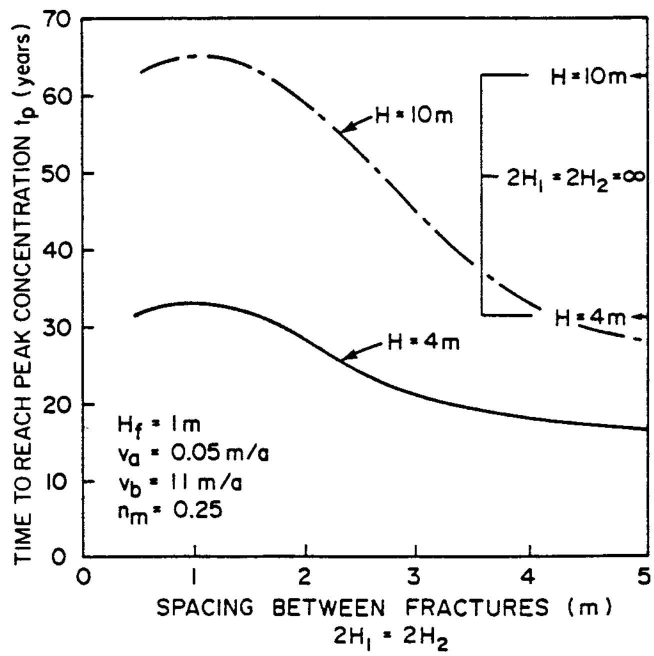


FIG. 7. Time required to reach peak impact on groundwater quality in an aquifer with fracture spacing.

for transport into the aquifer. In these analyses, the mass is represented in terms of the equivalent height of leachate, H_f. The determination and significance of this parameter is discussed in detail by Rowe (1988) and will not be repeated herein. Suffice it to say that the greater the mass of contaminant (per unit area of landfill) available for transport into the hydrogeologic system (i.e., the greater the value of H_f), the longer the contaminating life span (all other things being equal) and the greater the potential impact on the underlying aquifer.

Close inspection of the curves given in Figs. 2 and 5 shows a rather curious effect of varying the fracture spacing. For example, in Fig. 2, it is seen that increasing the fracture spacing from 1 to 2 m results in a decrease in both the time at which the peak impact occurs and the magnitude of the peak impact. However, further increasing the fracture spacing from 2 to 5 m results in an increase in peak concentration, although the time at which it occurs is further reduced.

Figures 6 and 7 summarize the values of peak impact (c_p) and the time required to reach this peak impact (t_p) respectively, for a range of fracture spacings and two layer thicknesses. Also indicated are the values of c_p and t_p calculated assuming no fractures (i.e., $2H_1 = 2H_2 = \infty$). Increasing the fracture spacing from a value of 0.5 m results

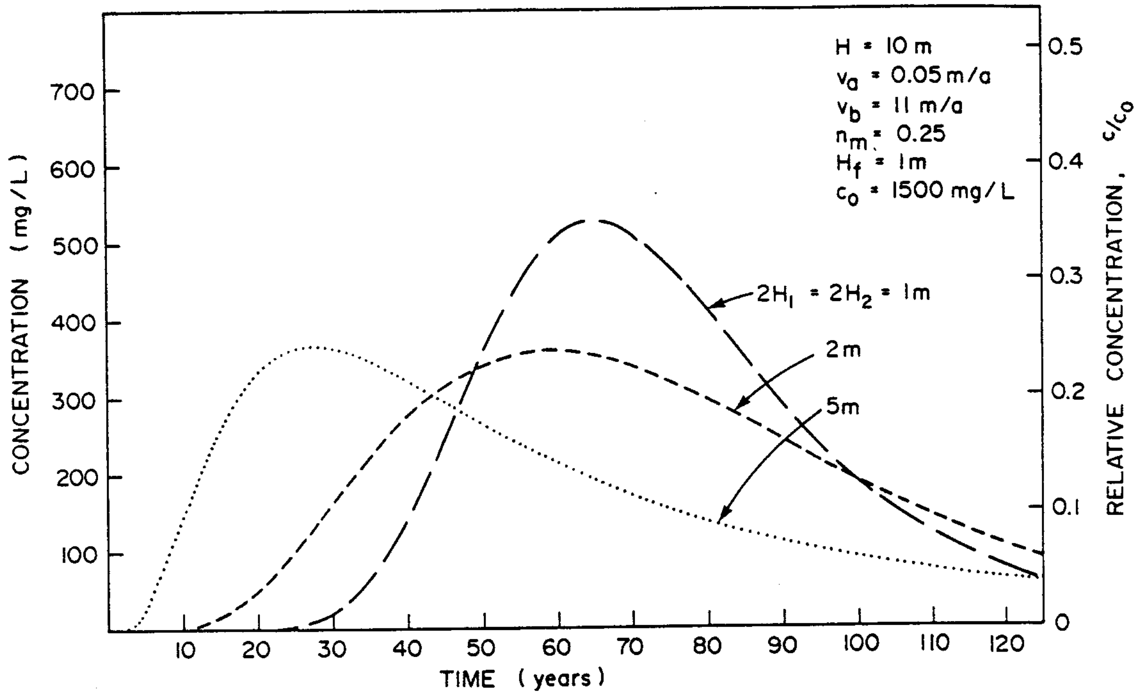


FIG. 5. Variation in concentration in the aquifer with time for a 10-m-thick fractured layer.

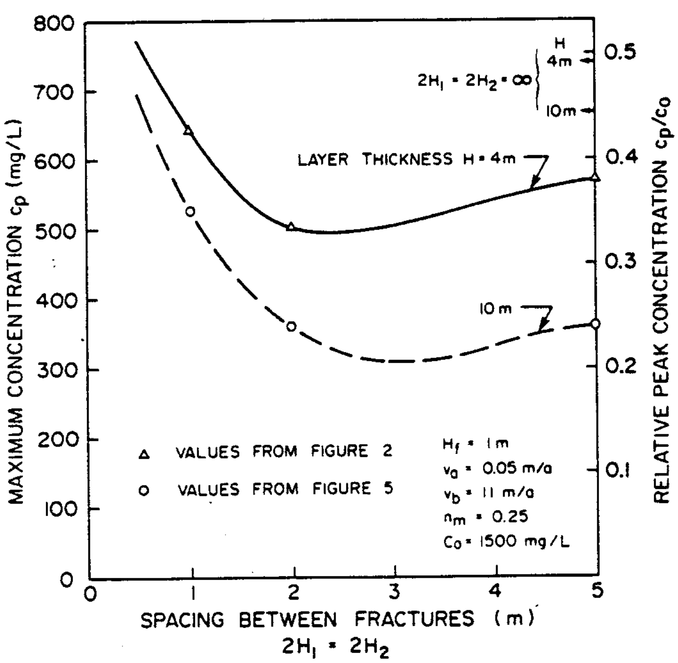
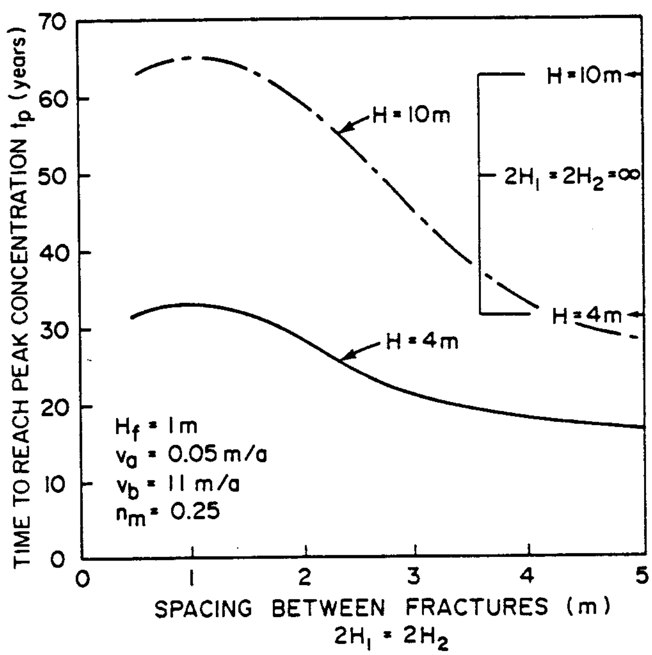


FIG. 6. Variation in peak impact on groundwater quality in an aquifer with fracture spacing.



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The behaviour evident in Fig. 6 can be explained in terms of the different levels of matrix diffusion that can occur. When the fracture spacing is very close (e.g., 1 m or less), there can be extensive and rapid movement of contaminant into the matrix pores between fractures. This slows down the movement of contaminant into the aquifer initially. However, once the concentration in the pores reaches the magnitude of the concentration in the fracture, the contaminant is no longer being slowed by matrix diffusion and so it moves rapidly along the fractures and into the underlying aquifer. The consequent high level of mass loading on the aquifer results in significant peak impacts being calculated.

As the fracture spacing is increased, there is a slower, more controlled movement of contaminants into the matrix (i.e., it takes longer to saturate the pores between fractures with contaminant). This allows earlier release of contaminant into the aquifer, but since the actual mass being released, per unit time, is reduced, dilution in the aquifer can have a greater effect. Thus the peak impact is reduced.

Since the time required to "saturate" the pores between fractures is related to the spacing of the fractures (all other things being equal), there is a spacing at which this time becomes large compared with the time required for contaminant to move down through the fractures to the underlying aquifer. This time is related to the thickness of the deposit (i.e., the thicker the deposit, the longer it takes for contaminant to reach the aquifer and, hence, the greater the spacing can be between fractures before the time required to saturate the pores becomes too large for matrix diffusion to have the greatest impact). This is why the spacing at which minimum peak impact is calculated for a 4-m-thick deposit (viz., $2H_1 = 2H_2 = 2.3$ m) is less than that for a 10-m-thick deposit (viz., $2H_1 = 2H_2 = 3$ m), although clearly the relationship is nonlinear. Once the spacing of fractures exceeds the optimal value (for a given layer thickness), the decreased effectiveness of matrix diffusion, resulting from the fact that there is not enough time for contaminant to diffuse fully into the matrix, gives rise to greater release of contaminant into the aquifer and, hence, greater peak impact.

It should be emphasized again that an implicit assumption in the theory presented in this paper is that contaminant migration from the landfill is through the fractures and not by advective–diffusive transport through the matrix between fractures. The assumption is reasonable, while the fracture spacing is less than the thickness of the deposit. However, the validity of this assumption decreases as the spacing between fractures approaches or exceeds the thickness of the deposit. In these cases, it would be prudent to perform analyses for both a fractured and unfractured medium. For the case considered here, the fractured analysis provides a conservative estimate of when first impact and peak impact will occur, but it may underestimate the maximum impact (e.g., see Figs. 6 and 7). If both indicate unacceptable impact (as is the case in Figs. 6 and 7), then a redesign (e.g., a compacted clay liner or greater control on leachate mounding) is necessary. For situations where one or the other calculation indicates that the impact is marginally acceptable, engineering judgement is required; this judgement may dictate the need for a more sophisticated analysis (e.g., finite element) that models both movement through the matrix from the landfill and movement through the fractures.

Summary and conclusion

Recent research has indicated that clayey tills that do not show signs of significant weathering may be fractured to depths in excess of 10 m. These clayey tills are often thought of as providing good barriers to minimize migration from landfill sites into underlying aquifers. However, the question of what impact fracturing may have has not received any significant attention.

This paper has presented a simple technique to model 1-D contaminant transport through a layer of fractured media and into an underlying aquifer. The model allows consideration of the finite mass of contaminant within the landfill as well as dilution due to the lateral flow of water in the aquifer. The fracture system may be composed of one set of parallel fractures or two sets of orthogonal fractures. The model allows consideration of advective–dispersive migration along the fracture, matrix diffusion from the fractures into the intact porous media, sorption, and radioactive decay. The model can be readily implemented on a microcomputer.

A limited parametric study has been performed to illustrate the application of the theory. For the specific cases and range of parameters considered in this study, the following is concluded.

(*a*) Even if the bulk hydraulic conductivity is known, variations in fracture spacing will have a significant effect on the time at which contaminant impact occurs within an underlying aquifer.

(*b*) The interaction between matrix diffusion, fracture spacing, and dilution in the aquifer can be readily quantified.

(*c*) The value of the effective matrix porosity used in the analysis can significantly affect the magnitude and time of occurrence of the impact on the aquifer, even if contaminant transport is through fractures.

(*d*) For a given fracture spacing, increasing the thickness of the clayey till between the landfill and the aquifer increases the time required for a given impact to reach the aquifer and decreases the maximum impact. However, for practical ranges of fracture spacing, significant impact on the aquifer is possible, even if the fractured clayey till is up to 10 m thick. Clearly, the Darcy velocity is a critical parameter in modelling contaminant migration through the fracture sytem (e.g., see also Rowe and Booker 1989*a*, *b*).

(*e*) Although not varied in the parametric study, varying the downward Darcy velocity, the equivalent height of leachate, the matrix diffusion coefficient, the aquifer base velocity, and sorption will all have a significant effect on calculated impact. The effect of uncertainty regarding these parameters can be readily examined using the proposed theoretical formulation.

Acknowledgement

Funding of the general programme of research into contaminant migration, of which this paper forms a part, has been provided by the Natural Sciences and Engineering Research Council of Canada (NSERC) under grant no. A1007. The computations reported in this paper were performed on a microcomputer, which was purchased with a grant from The University of Western Ontario Foundation Inc. The senior author also wishes to acknowledge the value of the NSERC Steacie Fellowship in providing the funding and freedom to conduct this research project.

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Appendix

The solution for 1-D, 2-D, and 3-D diffusion into a unit of matrix has been given by Rowe and Booker (1990). For completeness, it is summarized here.

Consider a typical unit of the matrix. If advection in the matrix is neglected, then the matrix concentration, c_m, satisfies the equation

$$[\mathrm{A}1a] \quad n_m D_m \nabla^2 c_m = (n_m + \rho_m K_m)\left(\frac{\partial c_m}{\partial t} + \lambda c_m\right)$$

where n_m is the matrix porosity; D_m is the diffusion coefficient in the matrix; ρ_m is the dry density of the matrix; K is the distribution–partitioning coefficient for the matrix; and λ is the decay constant for the solute. The concentration on the surface of the matrix is equal to that in the fissure system, c_f, so

$$[\mathrm{A}1b] \quad \begin{array}{l} c_m = c_f \text{ on the surface of the matrix} \\ c_m = 0 \text{ initially} \end{array}$$

On taking a Laplace transform, we find

[A2] $$n_m D_m \nabla^2 \bar{c}_m = (n_m + \rho_m K_m)(s + \lambda)\bar{c}_m$$

where $\bar{c}_m = \bar{c}_f$ on the surface of the matrix.

Now

[A3] $$\bar{q} = -\frac{\int \bar{F}_{mn}\, dS_m}{\int dV_m}$$

where $-F_{mn}$ denotes the normal component of flux entering the matrix; $\int \bar{F}_{mn}\, dS_m$ is the integral along the surface S_m of the fracture, and $\int dV_m$ is an integral over the volume V_m of the matrix between fractures. It thus follows, using the divergence theorem, that

[A4] $$\bar{q} = \frac{\int n_m D_m \nabla^2 \bar{c}_m\, dV_m}{\int dV_m} = \frac{n_m R_m (s + \lambda) \int \bar{c}_m\, dV_m}{\int dV_m}$$

where

$$R_m = 1 + \frac{\rho_m K_m}{n_m}$$

is the retardation coefficient for the matrix. Referring to [A4] and [10], we find that $\bar{\eta}$ is given by

[A5] $$\bar{\eta} = n_m R_m \frac{\int \bar{c}_m\, dV_m}{\int \bar{c}_f\, dV_m}$$

Case 1

First, suppose there is only a single set of fissures (set 1); then because of the assumed conditions in the landfill, assume there is no variation of concentration with respect to y. Also, the variation of concentration is likely to be relatively slow along the fissures (x direction) when compared with the variation between adjacent fissures (z direction); thus to sufficient accuracy, the concentration within the fissures satisfies the equation

[A6*a*] $$D_m \frac{\partial^2 \bar{c}_m}{\partial z^2} = R_m (s + \lambda)\bar{c}_m$$

where

[A6*b*] $$\bar{c}_m = \bar{c}_f, \qquad z = \pm H_1$$

It is thus found that

[A7] $$\bar{c}_m = \bar{c}_f \frac{\cosh \mu z}{\cosh \mu H_1}$$

where

$$\mu^2 = \frac{R_m (s + \lambda)}{D_m}$$

Hence

$$\frac{1}{2H_1}\int_{-H_1}^{H_1} \bar{c}_m\, dz = \frac{\bar{c}_f}{\mu H_1} \frac{\sinh \mu H_1}{\cosh \mu H_1}$$

so that

[A8] $$\bar{\eta} = n_m R_m \frac{\tanh \mu H_1}{\mu H_1}$$

Alternatively, the concentration can be expanded in a Fourier cosine series and

[A9] $$\bar{c}_m = \bar{c}_f \left(1 + \sum_j \chi_j \cos \alpha_j z\right)$$

where

$$\alpha_j = \frac{(j - 0.5)\pi}{H_1}$$

and so

[A10] $$\bar{\eta} = n_m R_m \left(1 + \sum_j \chi_j \frac{\sin \alpha_j H_1}{\alpha_j H_1}\right)$$

Now substituting into [A2], we find that

[A11] $$\chi_j = \frac{\sin \alpha_j H_1}{\alpha_j H_1} \frac{(s + \lambda)}{\left(s + \lambda + \alpha_j^2 \frac{D_m}{R_m}\right)}$$

Hence,

[A12] $$\bar{\eta} = n_m R_m \left[1 - 2 \sum_j \frac{(s + \lambda)}{\left(s + \lambda + \alpha_j^2 \frac{D_m}{R_m}\right)} \times \frac{1}{(\alpha_j H_1)^2}\right]$$

Case 2

Suppose that there are two sets of fissures (sets 1 and 2); then considering diffusive transport into the matrix in the x and y directions, the matrix concentration, $\bar{c}_m$, can be written in the form

[A13] $$\bar{c}_m = \bar{c}_f \left(1 + \sum_{j,k} \chi_{jk} \cos \alpha_j z \cos \beta_k y\right)$$

and

$$\alpha_j = (j - 0.5)\frac{\pi}{H_1}, \qquad \beta_k = (k - 0.5)\frac{\pi}{H_2}$$

so that

[A14] $$\bar{\eta} = n_m R_m \left(1 + \sum_{j,k} \chi_{jk} \frac{\sin \alpha_j H_1}{\alpha_j H_1} \frac{\sin \beta_k H_2}{\beta_k H_2}\right)$$

where

[A15] $$\chi_{jk} = \frac{\sin \alpha_j H_1}{\alpha_j H_1} \frac{\sin \beta_k H_2}{\beta_k H_2} \times \frac{(s + \lambda) R_m}{[R_m (s + \lambda) + D_m(\alpha_j^2 + \beta_k^2)]}$$

and so

[A16] $$\bar{\eta} = n_m R_m \left\{1 - 4 \sum_{j,k} \frac{(s + \lambda)}{\left[s + \lambda + \frac{D_m}{R_m}(\alpha_j^2 + \beta_j^2)\right]} \cdot \frac{1}{(\alpha_j H_1)^2} \cdot \frac{1}{(\beta_k H_2)^2}\right\}$$

It is possible to find a reasonably accurate approximation to [A16] for the case in which the spacing between the sets of orthogonal fissures are equal; viz, $H_1 = H_2$. In this case it seems reasonable to approximate diffusion into the square prism between adjacent fissures by diffusion into a cylinder of radius a having an equal cross-sectional area; that is, one for which

[A17] $a = 2H_1 / \sqrt{\pi}$

The solution for concentration within the prism is thus

[A18] $\bar{c} = c_f \dfrac{I_0\ (\mu r)}{I_0\ (\mu a)}$

where again

[A19] $\mu^2 = (s + \lambda) \dfrac{R_m}{D_m}$

and $I_0\ (\mu r)$ is a modified Bessel function of order zero. It then follows that

$$\text{[A20]} \quad \bar{\eta} = \frac{2n_m R_m}{a^2} \int_0^a I_0\ (\mu r) r\ dr$$

$$= 2n_m R_m \frac{{}_1\ (\mu a)}{\mu a}$$